Comment on "Liénard systems, limit cycles, and Melnikov theory"

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In papers by Sanjuán [Phys. Rev. E **57**, 340 (1998)] and Giacomini and Neukirch [Phys Rev. E **56**, 3809 (1997)] Liénard systems of the form $\dot{x}=y-\epsilon F(x,\mu)$, $\dot{y}=-x$ are studied. Sanjuán compares the results given by Melnikov theory with the results given by the R_n polynomials in the paper by Giacomini and Neukirch and conjectures that the roots of the R_n polynomials tend toward the roots of the Melnikov polynomial when $n\to\infty$, for arbitrary values of ϵ. We show here that this is true only when $\epsilon\to 0$ and that this fact strengthens the conjecture proposed by Giacomini and Neukirch. [S1063-651X(98)13112-4]

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For Liénard systems,

$$\dot{x}=y-\epsilon F(x,\mu),\qquad \dot{y}=-x, \tag{1}$$

the Melnikov function depends only on μ while the $R_n(x)$ polynomials depend on μ and ϵ. As pointed out in [1], Melnikov theory, as well as the R_n polynomials for Liénard systems, enables one to handle a global bifurcation problem by reducing it to an algebraic problem, that is, counting the number of roots of polynomials. In [1], the author conjectures that for a given Liénard system, there are associated a Melnikov polynomial $P(r^2)$ and two sequences of polynomials $R_n(x)$ and $g_{1,n}(x)$. For a fixed value of n, each positive root of $P(r^2)$ (α) is associated to a root of $R_n(x)$ (α_n) and to a root of $g_{1,n}(x)$ (β_n) such that $\alpha_n<\alpha<\beta_n$, and with the property that as n increases $\alpha_n\to\alpha$ and $\beta_n\to\alpha$.

Nevertheless, there is one major difference between the Melnikov method and the R_n method: the Melnikov method only works for $\epsilon\to 0$ while the R_n method is valid for all ϵ. In other words, the Melnikov theory is perturbative while the R_n method is not.

Hence, the conjecture presented at the end of [1] can only be true in the $\epsilon\to 0$ limit: one should find the same results with the R_n polynomials as with the Melnikov method, provided that $\epsilon\to 0$.

We give here two examples to illustrate this.

First we consider the van der Pol equation, that corresponds to system (1) with $F(x)=x^3/3-x$. Here, for all ϵ, the Melnikov polynomial $P(r^2)$ has $\alpha=2$ as root. If we take $\epsilon=3$, we find that for small n the root of the R_n polynomial (α_n) is increasing with n and is smaller than 2. But, calculating $R_{100}(x)$ and $R_{120}(x)$, we find $\alpha_{100}=2.006\ldots$ and $\alpha_{120}=2.008\ldots$ (with $R_{100}(\alpha_{100})<10^{-14}$ and $R_{120}(\alpha_{120})<10^{-21}$). Hence it is not true that $\alpha_n<\alpha$ for all n and it is not true that $\alpha_n\to\alpha$: α_n seems to tend toward $2.023\ldots$, which is the real maximum x value for the van der Pol limit cycle with $\epsilon=3$ (obtained from numerical integration).

Next we consider system (1) with $F(x)=x^5-\mu x^3+x$. For small ϵ, Melnikov theory tells us that for $\mu>\sqrt{\frac{40}{9}}$, there are two (circlelike) limit cycles of radii $\sqrt{\frac{3}{5}\mu\pm\frac{1}{5}\sqrt{9\mu^2-40}}$.

For example, let us take $\epsilon=\frac{1}{10}$ and $\mu=\sqrt{\frac{41}{9}}$. The Melnikov method predicts two (circlelike) limit cycles of radii: $r_1=1.039$ and $r_1=1.216$. The R_n polynomials have two positive roots of odd multiplicity. We see in Table I that for small ϵ the roots of the R_n polynomials tend to values very near those of the roots of the Melnikov function, as pointed out in [1].

However, if one takes $\epsilon=8$ and $\mu=\sqrt{\frac{41}{9}}$, Melnikov theory still predicts two (circlelike) limit cycles of the same

TABLE I. Values of the two roots of $R_n(x)$ for $\epsilon=\frac{1}{10}$ and $\mu=\sqrt{\frac{41}{9}}$.

n	2	4	6	8	10	20	30
Root 1	0.833	0.907	0.944	0.966	0.980	1.010	1.021
Root 2	1.199	1.191	1.189	1.189	1.191	1.197	1.202

TABLE II. Values of the two roots of $R_n(x)$ for $\epsilon=8$ and $\mu=\sqrt{\frac{41}{9}}$. For $n\geq 14$, there is no root any longer.

n	2	4	6	8	10	12	14	16
Root 1	0.83	0.89	0.94	0.97	1.01	1.05		
Root 2	1.19	1.19	1.17	1.15	1.13	1.09		

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radii (the Melnikov function does not depend on ϵ), while the R_n polynomials have no real root of odd multiplicity after $n=12$ (see Table II). The fact that the two real roots disappear indicates that there is no longer a limit cycle for $\epsilon=8$. Numerical integration shows that there is *no* limit cycle for $\epsilon=8$ and $\mu=\sqrt{\frac{41}{9}}$.

Although Melnikov theory is not effective at large ϵ, the R_n polynomials still give the right result.

[1] M. A. F. Sanjuán, Phys. Rev. E **57**, 340 (1998).